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PACKING RANDOM ITEMS OF THREE COLORS

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Consider three colors 1,2,3, and for $j \leq 3$, consider n items $(X_{i,j})_{i \leq n}$ of color j. We want to pack these items in n bins of equal capacity (the bin size is not fixed, and is to be determined once all the objects are known), subject to the condition that each bin must contain exactly one item of each color, and that the total item sizes attributed to any given bin does not exceed the bin capacity. Consider the stochastic model where the random variables $(X_{i,j})_{i \leq n, j \leq 3}$ are independent uniformly distributed over $[0,1]$. We show that there is a polynomial-time algorithm that produces a packing which has a wasted space $\leq K \log n$ with overwhelming probability.

1. Introduction

Consider m possible colors $1,\ldots,m$, and for $j \leq m$, consider n items $(x_{i,j})_{i \leq n}$. Let M be the matrix $(x_{i,j})_{i \leq n, j \leq m}$. What is the minimum bin size $a = a_{n,m}(M)$ needed to pack all the items, when the items are packed in n bins, subject to the restriction that each bin contains exactly one item of each color, and that the sum of the item sizes attributed to each bin does not exceed a? An equivalent reformulation of the problem is as follows. Consider a matrix $M = (x_{i,j})_{i \leq n, j \leq m}$ of positive numbers. Denote by Σ_n the set of permutations of $\{1,\ldots,n\}$. Consider the problem of finding

$$a_{n,m}(M) = \min \left\{ \max_{i \leq n} \sum_{j \leq m} x_{\sigma_j(i),j} \ ; \ \sigma_1,\ldots,\sigma_m \in \Sigma_n \right\}.$$

In words, we try to minimize the maximum row sum when we are permitted to permute elements within each column. Under this formulation, the problem has been studied under the name "assembly line crew scheduling" in [1], where it is pointed out that it is NP-complete, and where approximation algorithms are given.

The present paper is concerned with the probabilistic analysis of this problem. In our model the item sizes are independent random variables $(X_{i,j})_{i \leq n, j \leq m}$, uniformly distributed over $[0,1]$. The authors have, in the past, attempted to study stochastic bin packing models under the much less restrictive assumption that the item sizes are distributed according to a given distribution μ, on which no special assumption is made. It is most likely that such an analysis, along the lines of [3], could be made in the present situation. The point of focusing on the uniform distribution is that, in that case, some subtle phenomenon occur that do not occur in the general case.

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Since we have to pack items of total size $\sum_{\substack{i\leq n\\ j\leq m}} X_{i,j}$ in n bins, one of these bins must contain items of total size $\geq n^{-1} \sum_{\substack{i\leq n\\ j\leq m}} X_{i,j}$. Thus $a_{n,m}(M) \geq n^{-1} \sum_{\substack{i\leq n\\ j\leq m}} X_{i,j}$. It is natural to introduce the quantity

$$b_{n,m}(M) = a_{n,m}(M) - \frac{1}{n} \sum_{\substack{i\leq n\\ j\leq m}} X_{i,j}$$

that represents the average wasted (= empty) space in each bin in an optimal packing. The following question seems to be of unexpected difficulty.

Problem. Of which order is the expectation of $b_{n,m}(M)$?

The nature of the problem depends on the respective values of m, n. We are interested here in the case where m is fixed, $n \to \infty$.

Let us first consider the case $m=2$. In that case, the reader will check that the best we can do is to match the k-th largest item of the first column with the k-th smallest item of the second one. It is then routine to check that this implies that $Eb_{n,2}(M)$ is of order $n^{-1/2}$.

Things become much harder for $m \geq 3$. We conjecture that for given m, for n large, $Eb_{n,m}(M)$ is of order $n^{-h(m)}$ for some sequence $h(m) \to \infty$ ($h(m) = (m-1)/2$?). In this paper we prove the following.

Theorem 1. *For some constant K, we have*

$$Eb_{n,3}(M) \leq \frac{K \log n}{n}.$$

The point of this Theorem is that this bound is one order of magnitude smaller than the typical size $n^{-1/2}$ of the "random fluctuations". Our proof will show that a packing witnessing the inequality of Theorem 1 can be found in polynomial time. Minor modifications in our proof also show that $Eb_{n,m}(M) \leq K \log n/n$ for $m \geq n$; but in view of the conjecture above, the case $m=3$ is the most interesting. We do not know what is the actual order of $Eb_{n,3}(M)$. It is however possible to show that $Eb_{n,m}(M)$ is of order at least n^{-m+1} for m fixed, $n \to \infty$ (which, as the reader might have observed, is not sharp for $m=2$). We will not present the proof of this crude bound here. The main idea is that if we set

$$h(\varepsilon) = P\left(\max_{i\leq n} \sum_{j\leq m} X_{i,j} - \frac{1}{n} \sum_{\substack{i\leq n\\ j\leq m}} X_{i,j} \leq \varepsilon \right)$$

then $Eb_{n,m}(M) \geq \varepsilon/2$ whenever $(\mathrm{card}\Sigma_n)^{m-1} h(\varepsilon) \leq 1/2$. (One then estimates $h(\varepsilon)$.)

We now explain the main idea of our approach. One reformulation of the problem is as follows. We have to find a small value of a for which there is a solution to the following integer programming program. Find, for $i_1, i_2, i_3 \leq n$, numbers a_{i_1,i_2,i_3} that satisfy

$$a_{i_1,i_2,i_3} = 0 \text{ or } 1 \tag{1.1}$$

$$a_{i_1,i_2,i_3} \neq 0 \Rightarrow X_{i_1,1} + X_{i_2,2} + X_{i_3,3} \leq a \tag{1.2}$$

$$\sum a_{i_1,i_2,i_3} = 1, \tag{1.3}$$

where the summation is taken over any two indices, the third one being fixed. We will first study the linear programming problem where condition (1.1) is relaxed into $a_{i_1,i_2,i_3} \geq 0$. The main part of our proof is to show that when a is taken of order $n^{-1} \sum_{i \leq n} \sum_{j \leq 3} X_{i,j} + K \log n/n$, (where K is a universal constant) then with overwhelming probability the relaxed linear programming problem is feasible. We then show how to go from a solution to the relaxed problem to a solution of the original problem, by allowing a slight increase in a. A polynomial algorithm that actually finds a packing for which the average wasted space is $\leq K n^{-1} \log n$ can be found by solving the linear programming problem above and then following the explicit constructions described in section 2. It must be pointed out that the hardest arguments of the paper (sections 3 and 4) are irrelevant for the construction of this algorithm, and are only required to prove that the algorithm succeeds with overwhelming probability.

Here, as in the sequel, K denotes a universal constant, not necessarily the same at each occurrence. When we need to keep track of the constants involved, we denote them by $K_1, K_2, \ldots$.

For reasons that should become apparent later, it is more fruitful not to adopt the point of view of linear programming, but of a "continuous" version of it. Another way to look at the linear programming problem presented above is a follows: find a probability measure μ supported by the set

$$S_a = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \; ; \; \sum_{j \leq 3} x_j \leq a \right\}$$

such that for $j \leq 3$, the j-th marginal μ_j of μ is equal to $n^{-1} \sum_{i \leq n} \delta_{X_{i,j}}$, where δ_x denotes the unit mass concentrated at x. Not much seems to be known about measures with given marginals. Not surprisingly, the main step in the proof of Theorem 1 will be a theorem (that seems to be of independent interest) about measures on $[0,1]^3$ with given marginals.

We recall that for a signed measure μ, we define its total variation norm by

$$\|\mu\| = \sup \left\{ \left| \int f d\mu \right| \; ; \; f \text{ continuous}, |f| \leq 1 \right\}.$$

Throughout the paper, λ denotes the uniform probability on $[0,1]$.

Theorem 2. *There exists a number $\eta > 0$ with the following property. Consider for $j \leq 3$ a probability measure μ_j, on $[0,1]$ such that $\|\mu_j - \lambda\| \leq \eta$. Let $a = \sum_{j \leq 3} \int_0^1 x d\mu_j(x)$. Set*

$$S_a = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \; ; \; \sum_{j \leq 3} x_j \leq a \right\}.$$

Then there is a probability measure μ supported by S_a that has the measures μ_j as marginals.

To understand this result, one should observe that, whenever μ has measures μ_j as marginals, then

$$\int (x_1+x_2+x_3)d\mu(x_1,x_2,x_3) = \sum_{j\le 3}\int_0^1 x_j d\mu_j(x_j) = a.$$

Thus, if one requests that $\sum_{j\le 3} x_j \le a$ μ a.e., then we will actually have $x_1+x_2+x_3 = a$ μ a.e. (and it can't be required that μ be supported by $S_{a'}$ for $a'<a$). Another important observation is that Theorem 2 would not hold in dimension 2. (This is closely connected to the specific behavior mentioned earlier in the case $m=2$). To see it, it suffices to choose μ_1,μ_2 such that for $j=1,2$, we have $\int_0^1 x d\mu_j(x) = 1/2$ but $\mu_j(]1/2,1]) > 1/2$. Thus, if μ has μ_1,μ_2 as marginals, we have

$$\mu(]1/2,1]\times[0,1]) > 1/2; \quad \mu([0,1]\times]1/2,1]) > 1/2$$

and hence

$$\mu(]1/2,1]\times]1/2,1]) > 0$$

so that μ cannot be supported by the set $\{(x_1,x_2); x_1+x_2\le 1\}$.

2. Proof of Theorem 1

Our first result is basic to our approach. It will allow to use measures with given marginals to construct actual matchings. In the sequel, we assume $n\ge 2$.

Proposition 2.1. *There exists a universal constant K_1 with the following property. Consider a probability measure μ on $\mathbb{R}^3$, and its marginals $\mu_j, j=1,2,3$. Assume that for some closed set $S\subset\mathbb{R}^3$ we have $\mu(S)=1$. Given $n\ge 2$, we can find for $i\le n$ points $(z_j^i)_{j\le 3}$ in S such that*

$$\forall j\le 3, \forall t\in\mathbb{R}, n\mu_j([t,\infty)) \le \operatorname{card}\{i\le n\,;\, z_j^i\ge t\} + K_1\log n.$$

Comment. The important fact is that $z^i\in S$. Proposition 2.1 will later be applied when S is of the type $\{z; z_1+z_2+z_3\le a\}$. In that case $\sum_{j\le 3} z_j^i \le a$ for all $i\le n$.

Proof. The proof is very similar to that of [2], Theorem 2. It relies on the following fact, called Caratheodory Theorem: if a point in $\mathbb{R}^m$ is in the convex hull of a set, it can be expressed as a convex combination of at most $m+1$ points of that set.

For $j\le 3$, we set $u_j^0=\infty$, and we define by induction

$$u_j^{k+1} = \sup\{u<u_j^k\,;\, n\mu_j([u,\infty)) \ge n\mu_j([u_j^k,\infty))+1\}.$$

We observe that

$$n\mu_j([u_j^{k+1},\infty)) \geq n\mu_j([u_j^k,\infty)) + 1.$$

Thus the construction stops at a last $u_j^{k(j)}$, and $k(j) \leq n$.

For each $u \in \mathbb{R}$, each $j \leq 3$, we have

$$\mu_j([u,\infty)) = \int_S 1_{[u,\infty)}(z_j)d\mu(z_1,z_2,z_3).$$

The integrals of finitely many functions can simultaneously by approximated by finite sums. Thus we can find a finite set $L \subset S$, and for $z \in L$ a number $\alpha_z \geq 0$, such that $\sum_{z\in L} \alpha_z = n$, and that for $j \leq 3, 1 \leq k \leq k(j)$,

$$\begin{aligned} n\mu_j([u_j^k,\infty)) &\leq 1 + \sum_{z\in L} \alpha_z 1_{[u_j^k,\infty)}(z_j) \\ &= 1 + \sum_{z\in L} \alpha_z 1_{(-\infty,z_j]}(u_j^k). \end{aligned}$$

We have $k(1)+k(2)+k(3) \leq 3n$ points u_j^k. From Caratheodory Theorem, we see that there is a subset $M \subset L$ with $\mathrm{card} M \leq 3n+1$ and for $z \in M$ numbers $\beta_z, \beta_z > 0$, $\sum_{z\in M} \beta_z = \sum_{z\in L} \alpha_z = n$ such that for $j \leq 3, k \leq k(j)$ we have

$$\sum_{z\in M} \beta_z 1_{(-\infty,z_j]}(u_j^k) = \sum_{z\in L} \alpha_z 1_{(-\infty,z_j]}(u_j^k)$$

and thus,

$$n\mu_j([u_j^k,\infty)) \leq 1 + \sum_{z\in M} \beta_z 1_{(-\infty,z_j]}(u_j^k).$$

For $0 \leq k \leq k(j)$ and $u_j^{k+1} < u \leq u_j^k$ (setting $u_j^{k(j)+1} = -\infty$) we have by definition of u_j^{k+1} that $n\mu_j([u,\infty]) < n\mu_j([u_j^k,\infty))+1$. Since $1_{(-\infty,z_j]}(u) \geq 1_{(-\infty,z_j]}(u_j^k)$, it follows that for $j \leq 3, u \in \mathbb{R}$, we have

$$n\mu_j([u,\infty)) \leq 2 + \sum_{z\in M} \beta_z 1_{(-\infty,z_j]}(u), \tag{2.2}$$

where $\mathrm{card} M \leq 3n+1$, $\sum_{z\in M} \beta_z = n$.

The rest of the proof will rely on an iteration procedure. The basic step is contained in the following lemma.

Lemma 2.2. *Let $k \geq 4$. Consider a set $M \subset \mathbb{R}^3$ with $\mathrm{card} M \leq 2^k$ and for $z \in M$ a number $\beta_z \geq 0$ with $\sum_{z\in M} \beta_z \leq 2^k$. Then there is a set $M' \subset M$ with $\mathrm{card} M' \leq 2^{k-1}$ and for $z \in M'$ a number $\gamma_z \geq 0$, such that $\sum_{z\in M'} \gamma_z = \sum_{z\in M} \beta_z$ and for $u \in \mathbb{R}, j \leq 3$*

$$\sum_{z\in M} \beta_z 1_{(-\infty,z_j]}(u) \leq \sum_{z\in M'} \gamma_z 1_{(-\infty,z_j]}(u) + 7. \tag{2.3}$$

Proof. Set $f_j(u) = \sum_{z\in M} \beta_z 1_{(-\infty,z_j]}(u)$. For $j \le 3$, define $u_j^0 = \infty$, and by induction define

$$u_j^{k+1} = \sup\{u < u_j^k \; ; \; f_j(u) \ge f_j(u_j^k) + 7\}.$$

Since $f_j(u_j^{k+1}) \ge f_j(u_j^k)+7$, this construction stops at a last point $u_j^{k(j)}$ for which $7k(j) \le f_j(u_j^{k(j)}) \le \sum_{z\in M} \beta_z \le 2^k$. Thus we have, since $k \ge 4$

$$1 + \sum_{j\le 3} k(j) \le 1 + \frac{3\cdot 2^k}{7} \le 2^{k-1}.$$

Using Caratheodory theorem, we find $M' \subset M$ with $\operatorname{card} M' \le 1 + \sum_{j\le 3} k(j) \le 2^{k-1}$, and for $z \in M'$ a number $\gamma_z \ge 0$, with $\sum_{z\in M'} \gamma_z = \sum_{z\in M} \beta_z$ such that for $j \le 3, k \le k(j)$ we have

$$f_j(u_j^k) = \sum_{z\in M'} \gamma_z 1_{(-\infty,z_j]}(u_j^k).$$

Inequality (2.3) then follows by the argument used to prove (2.2). ∎

We now finish the proof of Proposition 2.1. Denote by q the smallest integer such that $3n+1 \le 2^q$. Using (2.2) and Lemma 2.2, we see by decreasing induction over k that for $k \ge 4$, there exists sets $M_k \subset M$, with $\operatorname{card} M_k \le 2^k$, for $z \in M_k$ numbers $0 \le \beta_z^k \le 1$, and for $z \in M$ nonnegative integers m_z^k such that for $j \le 3, u \in \mathbb{R}$

$$n\mu_j([u,\infty)) \le 2 + 7(q-k) + \sum_{z\in M_k} \beta_z^k 1_{(-\infty,z_j]}(u) + \sum_{z\in M} m_z^k 1_{(-\infty,z_j]}(u) \tag{2.4}$$

and

$$n = \sum_{z\in M} m_z^k + \sum_{z\in M_k} \beta_z^k. \tag{2.5}$$

Indeed we apply Lemma 2.2 to the function $\sum_{z\in M_k} \beta_z^k 1_{(-\infty,z_j]}(u)$ and set $\beta_z^{k-1} = \gamma_z - \lfloor \gamma_z \rfloor$ for $z \in M_{k-1} = M', m_z^{k-1} = m_z^k + \lfloor \gamma_z \rfloor$ for $z \in M$. Consider now (2.4) for $k = 4$. Since $\beta_z^4 \le 1$, we have

$$\begin{aligned} n\mu_j([u,\infty)) &\le 2 + 7(q-4) + 2^4 + \sum_{z\in M_k} m_z^4 1_{(-\infty,z_j]}(u) \\ &\le 7q - 10 + \sum_{z\in M} m_z^4 1_{(-\infty,u]}(z_j). \end{aligned}$$

Since $\sum_{z\in M} m_z^4 \le n$ by (2.5), this implies the result. (The collection $(z^i)_{i\le n}$ of points of S consists of the points z of M, the point z being counted with multiplicity m_z^4.) ∎

The empirical measure $\frac{1}{n}\sum\limits_{i\le n}\delta_{X_{i,j}}$ is always at distance 2 (for the total variation norm) from λ. In order to be able to apply Theorem 2, we will have to introduce an auxiliary measure close to λ, that suitably controls the random sequence $(X_{i,j})_{i\le n}$. We recall that η is introduced in the statement of Theorem 1, and K_1 in the statement of Proposition 2.1. We can and do assume $\eta\le 1/2$. Consider a parameter $h\ge 4K_1$. Set $q=q_n=\lfloor n/(h\log n)\rfloor$. Thus $n/q\ge 4K_1\log n$. For $\ell\le q$, set $I_\ell=[(\ell-1)/q,\ell/q[$.

Lemma 2.3. *Consider $x_1,\dots,x_n\in[0,1[$. Assume that for $1\le\ell\le q$, we have*

$$\frac{n}{q}\left(1-\frac{\eta}{2}\right)\le \operatorname{card}\{i\le n\ ;\ x_i\in I_\ell\}\le\frac{n}{q}\left(1+\frac{\eta}{2}\right). \tag{2.6}$$

Then for $n\ge n_h$ (where n_h depends on h only) there exists a probability measure μ on $[2/q,1]$ with the following properties

$$\forall u,\ \frac{2}{q}\le u\le 1, n\mu([u,1])\ge\operatorname{card}\{i\le n\ ;\ x_i\ge u\}+K_1\log n \tag{2.7}$$

$$n\int_0^1 td\mu(t)\le\sum_{i\le n}x_i+Kh\log n. \tag{2.8}$$

$$\|\mu-\lambda\|\le\eta. \tag{2.9}$$

Proof. Define the probability μ as follows. For $2\le\ell\le q$, μ has mass $n^{-1}\operatorname{card}\{i\le n\,;\,x_i\in I_\ell\}$ uniformly spread in I_ℓ and has mass k/n concentrated at 1, where $k=\operatorname{card}\{i\le n\,;\,x_i\in I_1\cup I_2\}$. Consider $2/q\le u\le 1$, let ℓ be the largest integer such that $\ell/q\le u$. Then

$$n\mu([u,1])\ge k+\operatorname{card}\{i\le n\ ;\ x_i\in I_{\ell+1}\cup\ldots\cup I_n\}.$$

On the other hand

$$\operatorname{card}\{i\le n\ ;\ x_i\ge u\}\le\operatorname{card}\{i\le n\ ;\ x_i\in I_\ell\cup\ldots\cup I_n\}.$$

Thus (2.7) holds since, from (2.6) we have

$$K_1\log n+\operatorname{card}\{i\le n\ ;\ x_i\in I_\ell\}\le K_1\log n+\frac{5}{4}\frac{n}{q}\le\frac{6}{4}\frac{n}{q}\le k.$$

For $2<\ell\le n$, we have

$$\int_{I_\ell} td\mu(t)\le\mu(I_\ell)\left(\frac{\ell}{q}\right)\le\frac{\mu(I_\ell)}{q}+\mu(I_\ell)\left(\frac{\ell-1}{q}\right)$$
$$\le\frac{\mu(I_\ell)}{q}+n^{-1}\sum_{x_i\in I_\ell}x_i$$

and thus

$$\sum_{\ell\le n}\int_{I_\ell}td\mu(t)\le\frac{1}{q}+n^{-1}\sum_{i\le n}x_i$$

and hence

$$n\int_0^1 t d\mu(t) \leq \sum_{i\leq n} x_i + \frac{n}{q} + k \leq \sum_{i\leq n} x_i + Kh\log n.$$

This proves (2.8). It is simple to see that

$$\begin{aligned}\|\mu-\lambda\| &\leq \frac{2}{q} + \frac{k}{n} + \sum_{\ell=3}^{n}\left|\frac{1}{n}\mathrm{card}\{i\leq n\ ;\ x_i\in I_\ell\} - \frac{1}{q}\right| \\ &\leq \frac{Kh\log n}{n} + \eta/2 \leq \eta\end{aligned}$$

for n large enough. This proves (2.9). ∎

The following is a consequence of well known bounds on the tail of the binomial law.

Lemma 2.4. *Consider a sequence $X_1,\ldots,X_n$ i.i.d uniform over $[0,1]$. Then the following event*

$$\forall \ell\leq q,\ \frac{n}{q}\left(1-\frac{\eta}{2}\right) \leq \mathrm{card}\{i\leq n\ ;\ X_i\in I_\ell\} \leq \frac{n}{q}\left(1+\frac{\eta}{2}\right)$$

has probability $\geq 1-n^{1-h/K_2}$.

We now prove Theorem 1. Consider i.i.d uniform r.v. $(X_{i,j})_{i\leq n, j\leq 3}$. It follows from Lemmas 2.3 and 2.4 that with probability $\geq 1-3n^{1-h/K_2}$, there exists three probability measures μ_j, $1\leq j\leq 3$, supported by $[2/q,1]$, with the following properties

$$\forall u,\ \frac{2}{q}\leq u\leq 1, \forall j\leq 3, \tag{2.10}$$
$$n\mu_j([u,1]) \geq \mathrm{card}\{i\leq n\ ;\ X_{i,j}\geq u\} + K_1\log n$$

$$\|\mu_j - \lambda\| \leq \eta \tag{2.11}$$

$$\sum_{j\leq 3}\int_0^1 t d\mu_j(t) \leq a =: \frac{Kh\log n}{n} + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{3} X_{i,j}. \tag{2.12}$$

We now show that under conditions (2.10) to (2.12), the items $X_{i,j}$ can be packed in bins of size a. To that aim, we first apply Theorem 2 to find a probability measure μ on $\mathbb{R}^3$ with marginals μ_j, and supported by the set $\{z\in\mathbb{R}^3\ ;\ z_1+z_2+z_3\leq a\}$. We observe that since $\mu_j([2/q,1])=1$, μ is actually supported by the set

$$S = \left\{z\in\mathbb{R}^3\ ;\ z_1,z_2,z_3\geq\frac{2}{q}\ ;\ z_1+z_2+z_3\leq a\right\}.$$

We then apply Proposition 2.1 to find n points $z^1,\ldots,z^n\in S$ such that for $j\leq 3$, $u\geq 2/q$, we have

$$n\mu_j([u,1]) \leq \mathrm{card}\{i\leq n\ ;\ z_j^i\geq u\} + K_1\log n.$$

Comparing with (2.10), we see that for $u \geq 2/q$ we have

$$\operatorname{card}\{i \leq n\,;\, X_{i,j} \geq u\} \leq \operatorname{card}\{i \leq n\,;\, z_j^i \geq u\}. \tag{2.13}$$

Since $z_j^i \geq 2/q$, it follows that (2.13) actually holds for all u. It follows from (2.13) that there exists a permutation σ_j of $\{1,\dots,n\}$ such that $X_{\sigma_j(i),j} \leq z_j^i$. Thus we have

$$\sum_{j\leq 3} X_{\sigma_j(i),j} \leq \sum_{j\leq 3} z_j^i \leq a,$$

and hence the items $X_{i,j}$ can be packed in bins of size $\leq a$. Thus we have shown that with probability $\geq 1-n^{1-h/K_2}$, we have $b_{n,3}(M) \leq Kh\log n/n$. Since, obviously, $b_{n,3}(M) \leq 3$, Theorem 1 follows by taking $h = \max(4K_1, 3K_2)$.

3. Proof of Theorem 2

It might not even be obvious to the reader that there exists a probability measure μ_0 on $\mathbb{R}^3$, having its three marginals equal to λ, and supported by the set $S_{3/2} = \{(x_1,x_2,x_3)\,;\, x_1+x_2+x_3 \leq 3/2\}$. Our first task will be the explicit construction of such a measure. This measure will actually play an important rôle in the proof of Theorem 2.

Consider the set

$$B = \left\{(x_1,x_2)\,;\ 0 \leq x_1,\ x_2 \leq 1\,;\ \frac{1}{2} \leq x_1+x_2 \leq \frac{3}{2}\right\}.$$

Proposition 3.1. *There exists a probability measure ν_0 on B with the following properties*

(3.1) *The two marginals of ν_0 are λ. The image of ν_0 under the map $(x_1,x_2) \to x_1+x_2$ is uniform on $[1/2,3/2]$.*

(3.2) *ν_0 has a density $\geq 1/K_3$ with respect to the two dimensional Lebesgue measure.*

Thus if we denote by μ_0 the image of ν_0 under the map $(x_1,x_2) \to \left(x_1,x_2,\frac{3}{2}-x_1-x_2\right)$, it is supported by $S_{3/2}$, and its 3 marginals are equal to λ.

Proof. For a measure ν on B, denote by $r(\nu)$ its image under the map $(x_1,x_2) \to x_1+x_2$. For $0 \leq u \leq 1/2$, consider the boundary R_u of the rectangle of vertices $(0,1-u),(u,1),(1,u),(1-u,0)$ and the probability γ_u uniformly spread on R_u. Then the two marginals of γ_u equal λ, while $r(\gamma_u)$ consists of two masses of $\frac{1}{2}(1-u)$ concentrated at $1-u$ and $1+u$ respectively, and a mass u uniformly spread on the interval $1-u, 1+u$. Set $h(u) = (1-u)^{-2}$, so that $\int_0^{1/2} h(u)du = 1$. Consider the following mixture of the probabilities γ_u:

$$\nu_0 = \int_0^{1/2} h(u)\gamma_u du.$$

Obviously its two marginals are equal to λ, and (3.2) holds. (The density of ν_0 could easily be computed, but its actual value is irrelevant for our purposes). It remains only to show that

$$r(\nu_0) = \int_0^{1/2} h(u) r(\gamma_u) du$$

is uniform on [1/2,3/2]. Obviously $r(\nu_0)$ is symmetric around 1. For $0 \le t \le 1/2$ we have

$$r(\nu_0)([1, 1+t]) = \frac{1}{2} \int_0^t h(u) du + \frac{1}{2} \int_t^{1/2} \frac{t}{u} \cdot u \cdot h(u) du.$$

Elementary computations show that this is t. This finishes the proof. ∎

Our proof of Theorem 2 will be non-constructive. It uses the following criteria, due to V. Strassen. (The proof of this criteria relies on the Hahn-Banach theorem).

Proposition 3.2. [4] *Consider three probability measures μ_j, $j = 1,2,3$ on $[0,1]$, and a closed set $S \subset [0,1]^3$. The following are equivalent*

(1) *There exists a probability measure μ, supported by S, of marginals $\mu_j, j = 1,2,3$.*
(2) *Given continuous functions ϑ_j, $j = 1,2,3$ on $[0,1]$ that satisfy*

$$(x_1, x_2, x_3) \in S \Rightarrow \sum_{j \le 3} \vartheta_j(x_j) \le 1 \tag{3.3}$$

then $\sum_{j \le 3} \int_0^1 \vartheta_j d\mu_j \le 1$.

While Proposition 3.2 is easy to prove, the criteria it provides is by no means easy to use, the difficulty being to understand which triplets of functions satisfy (3.3). The main step toward this, in the case $S = S_a$, is the forthcoming Proposition 3.3. We denote by J the interval $\left[-\frac{1}{10}, \frac{11}{10}\right]$. The choice of 1/10 there is done for convenience and has no special meaning.

Proposition 3.3. *Consider two functions f_1, f_2 from J to $\mathbb{R}$. For $x \in J + J = [-2/10, 22/10]$, set*

$$g(x) = \sup\{f_1(x_1) + f_2(x_2) \,;\, x_1, x_2 \in J, x_1 + x_2 = x\}.$$

Let $\varepsilon = \int_B (g(x_1 + x_2) - f_1(x_1) - f_2(x_2)) dx_1 dx_2$. Then we can find functions f_1', f_2' from J to $\mathbb{R}$ and real numbers a_1, a_2, α with the following properties

$$f_i' \ge f_i \text{ for } i = 1,2 \tag{3.4}$$

$$\text{For } x \in \left[\frac{4}{10}, \frac{16}{10}\right] = \frac{3}{2} - J, x_1, x_2 \in J, x_1 + x_2 = x, \tag{3.5}$$
$$\text{we have } g(x) \ge f_1'(x_1) + f_2'(x_2)$$

$$\text{For } i = 1,2, \text{ and } x \in J, \text{ we have } |f_i'(x) - (a_i + \alpha x)| \le K_4 \varepsilon. \tag{3.6}$$

The proof of that result, while purely elementary, is somewhat lengthy and uninspiring. Thus, in order not to discourage the reader, we postpone this proof to section 4, and we turn toward the nicest part of the proof of Theorem 2, that is the deduction of that theorem from Proposition 3.3.

For simplicity, we will say that a triple of functions $(f_j)_{j\leq 3}$ from J to $\mathbb{R}$ is *admissible* if it satisfies the following condition:

$$\sum_{j\leq 3} f_j(x_j) \leq 1 \text{ whenever } x_1, x_2, x_3 \in J,\ \sum_{j\leq 3} x_j = 3/2.$$

We say that an admissible triple $(f_j)_{j\leq 3}$ is *maximal* if whenever we have an admissible triple $(f'_j)_{j\leq 3}$ such that $f'_j \geq f_j$ for $j\leq 3$, then $f_j = f'_j$ for $j\leq 3$.

The following is a crucial consequence of Proposition 3.3.

Proposition 3.4. *Consider a maximal admissible triple of functions $(f_j)_{j\leq 3}$ from J to $\mathbb{R}$. Let $\varepsilon = 1-\sum_{j\leq 3}\int f_j d\lambda$. Then there exists $(a_j)_{j\leq 3}$ and α such that, for $j\leq 3$ and $x\in J$ we have*

$$|f_j(x) - (a_j + \alpha x)| \leq K_5\varepsilon.$$

Proof. For $x\in[-2/10, 22/10]$, we define

$$g(x) = \sup\{f_1(x_1) + f_2(x_2)\ ;\ x_1, x_2 \in J, x_1 + x_2 = x\}.$$

Since the triplet $(f_j)_{j\leq 3}$ is admissible, we have $g\left(\frac{3}{2} - x\right) + f_3(x) \leq 1$ whenever $x \in J$. It follows that $\int_{1/2}^{3/2} g(x)dx \leq 1 - \int f_3 d\lambda$. We now recall the measure ν_0 constructed earlier. We have

$$\begin{aligned}&\int_B (g(x_1 + x_2) - f_1(x_1) - f_2(x_2))d\nu_0(x_1, x_2)\\ &= \int_{1/2}^{3/2} g(x)dx - \int f_1 d\lambda - \int f_2 d\lambda \leq 1 - \sum_{j\leq 3}\int f_j d\lambda = \varepsilon.\end{aligned}$$

From (3.2) it follows that

$$\int_B (g(x_1 + x_2) - f_1(x_1) - f_2(x_2))dx_1 dx_2 \leq K_3\varepsilon.$$

From Proposition 3.3 we find $f'_i \geq f_i$ for $i = 1, 2$ and a_1, a_2, α that satisfy (3.5) and (3.6) (with ε replaced by $K_3\varepsilon$).

Given $x_1, x_2, x_3 \in J, x_1+x_2+x_3 = 3/2$, we have $3/2 - x_3 \in 3/2 - J$, so that, by (3.5)

$$f'_1(x_1) + f'_2(x_2) \leq g\left(\frac{3}{2} - x_3\right) \leq 1 - f_3(x_3).$$

This shows that the triple (f_1', f_2', f_3) is admissible. Since we supposed that $(f_j)_{j\le 3}$ is maximal, we must have $f_i' = f_i$, $i=1,2$. Thus from (3.6) follows that

$$|f_i(x) - (a_i + \alpha x)| \le K_3 K_4 \varepsilon \text{ for } x \in J, i = 1,2.$$

We can apply the same argument to the couple f_1, f_3 to find a_1', a_3', α' such that $|f_i(x) - (a_i' + \alpha' x)| \le K_3 K_4 \varepsilon$ for $i = 1,3$. Taking $i=1, x=0$ shows that $|a_1 - a_1'| \le 2K_3K_4\varepsilon$. Taking $x=1$ then shows that $\alpha - \alpha' \le 3K_3K_4$. Setting $a_3 = a_3'$, it then follows that $|f_j(x) - (a_j + \alpha x)| \le 6K_3K_4\varepsilon$ for $j \le 3$, $x \in J$. ∎

We now prove two simple lemmas.

Lemma 3.5. *Consider three bounded functions $\vartheta_j, j \le 3$ from $[0,1]$ to $\mathbb{R}$ and a number $b \ge 0$. Assume that*

$$x_1, x_2, x_3 \in [0,1], \sum_{j\le 3} x_j \le b \Rightarrow \sum_{j \le 3} \vartheta_j(x_j) \le 1.$$

Then we can find three functions g_j from $\mathbb{R}$ to $\mathbb{R}$ such that $g_j(x) = \vartheta_j(x)$ for $x \in [0,1]$, and $\sum_{j\le 3} g_j(x_j) \le 1$ whenever $\sum_{j \le 3} x_j \le b$.

Proof. Define $A = \max\left(0, \sum_{j\le 3} \sup\{\vartheta_j(x) : x \in [0,1]\}\right)$. We define $g_j(x) = \vartheta_j(x)$ if $x \in [0,1]$. We define $g_j(x) = \vartheta_j(1)$ if $x \ge 1$. We define $g_j(x) = \vartheta_j(0) - A$ if $x < 0$. Consider now x_1, x_2, x_3 such that $\sum_{j\le 3} x_j \le b$. We want to show that $\sum_{j \le 3} g_j(x_j) \le 1$. Clearly we can assume that $x_j \le 1$ for $j \le 3$. The result is obvious if $x_j \ge 0$ for $j \le 3$. But if $x_j < 0$ for some $j \le 3$, we actually have

$$\sum_{j \le 3} g_j(x_j) \le \sum_{j\le 3} \sup\{\vartheta_j(x) \; ; \; x \in [0,1]\} - A \le 0.$$ ∎

Lemma 3.6. *Consider an admissible triple $(h_j)_{j \le 3}$ of functions from J to $\mathbb{R}$. Then there exists an admissible triple $(f_j)_{j\le 3}$, $f_j \ge h_j$ for $j \le 3$, that is maximal.*

Proof. Define, for $x \in J$

$$f_1(x) = \inf\{1 - h_2(x_2) - h_3(x_3) \; ; \; x_2, x_3 \in J, \; x + x_2 + x_3 = 3/2\}.$$

Obviously the triple (f_1, h_2, h_3) is admissible, and $f_1 \ge h_1$ since $(h_j)_{j \le 3}$ is admissible. Also, if an admissible triple $(f_j')_{j\le 3}$ satisfies $f_j' \ge h_j$ for $j \le 3$, then $f_1' \le f_1$. Thus if $f_1' \ge f_1$ we must have $f_1' = f_1$. Thus we have replaced the admissible triple by one where the first function is maximal. We proceed in the same way for the second, and then the third function. ∎

We now come to the crucial point. In view of Proposition 3.2, Theorem 2 follows from the next result.

Proposition 3.7. *There exists $\eta > 0$ with the following property. Suppose that we are given three probability measures $(\mu_j)_{j\le 3}$ such that $\|\mu_j - \lambda\| \le \eta, j \le 3$. Let $b = \sum_{j\le 3} \int x d\mu_j(x)$. Consider three continuous functions ϑ_i from $[0,1]$ to $\mathbb{R}$ such that*

$$x_1, x_2, x_3 \in [0,1], \sum_{j\le 3} x_j = b \Rightarrow \sum_{j\le 3} \vartheta_j(x_j) \le 1. \tag{3.7}$$

Then $\sum_{j\le 3} \int \vartheta_j d\mu_j \le 1$.

Proof. We first use Lemma 3.5 to construct functions $(g_j)_{j\le 3}$ on $\mathbb{R}$ such that $g_j(x) = \vartheta_j(x)$ for $x \in [0,1]$ and

$$\sum_{j\le 3} x_j = b \Rightarrow \sum_{j\le 3} g_j(x_j) \le 1. \tag{3.8}$$

Define a by $b = 3/2 + 3a$. Since $|b - 3/2| \le 3\eta$, we have $|a| \le \eta$. We surely can assume $\eta \le 1/20$, so that $|a| \le 1/20$.

Consider the function h_j from J to $\mathbb{R}$ given by $h_j(x) = g_j(x+a)$. It follows from (3.8) that the triple $(h_j)_{j\le 3}$ is admissible. Using Lemma 3.6, we can find a maximal admissible triple $(f_j)_{j\ge 3}$, with $f_j \ge h_j$.

We set $\varepsilon = 1 - \sum_{j\le 3} \int f_j d\lambda$. According to Proposition 3.4, we can find numbers $(a_j)_{j\le 3}$ and α such that for $x \in J$,

$$|f_j(x) - (a_j + \alpha x)| \le K_5 \varepsilon. \tag{3.9}$$

Set $\xi_j(x) = f_j(x-a)$. For $x \in [0,1]$, we have $x - a \in J$, so

$$\xi_j(x) = f_j(x-a) \ge h_j(x-a) = g_j(x) = \vartheta_j(x).$$

Moreover, if λ_a denotes the uniform probability on $[a, 1+a]$, we have $\int f_j d\lambda = \int \xi_j d\lambda_a$. Also, from (3.9), we see (since $|a| \le 1/20$) that for $x \in [-1/20, 21/20]$ we have

$$|\xi_j(x) - (b_j + \alpha x)| \le K_5 \varepsilon, \tag{3.10}$$

where $b_j = a_j - \alpha a$. We have

$$\int \vartheta_j d\mu_j \le \int \xi_j d\mu_j = \int \xi_j d\lambda_a + \int \xi_j d\nu_j = \int f_j d\lambda + \int \xi_j d\nu_j, \tag{3.11}$$

where ν_j is the (signed) measure $\mu_j - \lambda_a$. From (3.10) we see that

$$\left| \int \xi_j d\nu_j - \int (b_j + \alpha x) d\nu_j(x) \right| \le K_5 \varepsilon \|\mu_j - \lambda_a\| \le 3K_5 \varepsilon \eta$$

since $\|\mu_j - \lambda_a\| \le \|\mu_j - \lambda\| + \|\lambda - \lambda_a\| \le \eta + 2a \le 3\eta$. Now

$$\int (b_j + \alpha x) d\nu_j(x) = \int (b_j + \alpha x) d\mu_j(x) - \int (b_j + \alpha x) d\lambda_a(x)$$
$$= b_j + \alpha \int x d\mu_j(x) - b_j - \alpha \left(\frac{1}{2} + a \right).$$

Thus

$$\int \vartheta_j d\mu_j \leq \int f_j d\lambda + \alpha \left(\int x d\mu_j(x) - \left(\frac{1}{2} + a \right) \right) + 3K_5 \eta\varepsilon.$$

We remember that

$$b = 3\left(\frac{1}{2} + a\right) = \sum_{j \leq 3} \int x d\mu_j(x).$$

Thus, if we assume $\eta \leq 1/9K_5$, we have, by definition of ε

$$\sum_{j\leq 3} \int \vartheta_j d\mu_j \leq \sum_{j \leq 3} \int f_j d\lambda + \varepsilon \leq 1.$$

This completes the proof.

4. Proof of Proposition 3.3

Before we start the proof, we explain the ideas underlying the approach. Consider the two sets

$$A = \{(x,y) \; ; \; x \in J, y \leq f_1(x)\}$$
$$B = \{(x,y) \; ; \; -x \in J, y \geq -f_2(-x)\}.$$

For $0 < a < 2$, we denote by B_a the translation of B to the right by a length a. Consider the smallest value $t(a)$ such that if one translates upwards the interior of B_a by $t(a)$, this interior does not meet the interior of A. It is a simple matter to see that $t(a) = g(a)$. Denote by C_a the translation upwards of B_a by $t(a) = g(a)$.

Consider the region

$$R(a) = \{(x,y) \; ; \; x \in [0,1], a - x \in [0,1], f_1(x) \leq y \leq g(a) - f_2(a-x)\},$$

and for $x_1, x_2 \in J$, set

$$h(x_1, x_2) = g(x_1 + x_2) - f_1(x_1) - f_2(x_2).$$

By hypothesis, we have $\varepsilon = \int_B h(x,y) dx dy$. It is a simple matter to see that this means that the average value of the area of $R(a)$, for $a \in [1/2, 3/2]$ is ε. This means that the upper boundaries of A and the lower boundary of C_a have approximately matching shapes for most values of a. A typical situation where this occurs is when $f_1(x) = a_1 + \alpha x - \tau_1(x), f_2 = a_1 + \alpha x - \tau_2(x)$, where $\tau_1, \tau_2 \geq 0$, and $\int_J \tau_1(x) dx, \int_J \tau_2(x) dx$ are small (and the proof will show that this special situation is actually the general case). There is a small proportions of points x for which $\tau_1(x)$ is much larger than its average. These points are not going to provide any information about the value of a_1 or α, and the first task is to identify these points. This is done by observing that, for such a point, the average value of $g(a) - f_1(x) - f_2(a-x)$ over a is abnormally large. More precisely, for $x \in [0,1]$, let $B_x = \{y \in [0,1] \, ; (x,y) \in B\}$. Set

$$h(x) = \int_{B_x} h(x,y) dy.$$

By Fubini Theorem, we have

$$\varepsilon = \int_B h(x,y)dxdy = \int_0^1 h(x)dx.$$

Let $U = \{x \in [0,1], h(x) \leq 50\varepsilon\}$. Thus $\lambda(U) \geq 49/50$. We can think to the points of U as typical points.

Lemma 4.1. *Consider $0 < b \leq 1/8$ and $a \in U$ such that $a+b \in U$, and consider the affine function ξ on $\mathbb{R}$ such that $\xi(a) = f_1(a), \xi(a+b) = f_1(a+b)$. Then, whenever $a+c \in J$ and $|c| \leq 1/8$, we have*

$$f_1(a+c) \leq \xi(a+c) + K\varepsilon\left(1 + \frac{|c|}{b}\right).$$

Comments. This will be used for $|c|$ of order b. This means that the graph of f_1 cannot make a "spike" above the graph of ξ. The reason is that, since $R(a)$ is of small area for most values of a, such a spike must correspond for many values of a, to a "dent" in the boundary of C_a; but when a varies, these dents will combine to create a large hole in the boundary of B, and this large hold will prevent a close contact between the boundary of A and that of C_a. The quantification of this simple idea is unfortunately very uninspiring.

Proof. By definition of h, for $x_1, x_2 \in J$ we have

$$f_2(x_2) = g(x_1 + x_2) - f_1(x_1) - h(x_1, x_2).$$

Taking $x_1 = a, x_2 = y - a$, we have, for $y - a \in J$

$$f_2(y-a) = g(y) - f_1(a) - h(a, y-a). \tag{4.1}$$

Replacing a by $a+b$, we have, for $y-a-b \in J$ that

$$f_2(y-a-b) = g(y) - f_1(a+b) - h(a+b, y-a-b). \tag{4.2}$$

Replacing a by $a+c$, we have, for $y-a-c \in J$ that

$$f_2(y-a-c) = g(y) - f_1(a+c) - h(a+c, y-a-c)$$

and hence

$$f_2(y-a-c) \leq g(y) - f_1(a+c). \tag{4.3}$$

We set $\delta = f_1(a+c) - \xi(a+c)$. (This is the quantity that we want to bound.)

The function ξ is affine, thus of the type $\xi(t) = \alpha t + \beta$. Thus $\xi(a-y) = \xi(a) - \alpha y = f_1(a) - \alpha y$. We set, for $y - a \in J$:

$$f(y) = f_2(y-a) + \xi(a-y) = f_2(y-a) + f_1(a) - \alpha y.$$

We set $g'(y) = g(y) - \alpha y, h_1(y) = h(a, y-a)$. From (4.1) we see that, for $y - a \in J$, we have

$$f(y) = g'(y) - h_1(y). \tag{4.4}$$

For $y-a-b \in J$, we set $h_2(y) = h(a+b, y-a-b)$. From (4.2) we see that for $y-a-b \in J$, we have, since $f_1(a)+\alpha b = \xi(a+b) = f_1(a+b)$,

$$f(y-b) = g'(y) - h_2(y). \tag{4.5}$$

Since $\xi(a+c-y) = -\alpha y + \xi(a+c) = -\alpha y + f_1(a+c) - \delta$, we see from (4.3) that for $y-a-c \in J$, we have

$$f(y-c) \leq g'(y) - \delta. \tag{4.6}$$

To understand the point of (4.4) to (4.6), the reader could try first the case $h_1 = h_2 = 0$. In that case, $f(y-b) = f(y)$ and $f(y-c) \leq f(y) - \delta$ for many values of y, which would imply $\delta \leq 0$. It is unfortunately not true that h_1 and h_2 are zero; but these functions have a small integral. Since $a \in U, a+b \in U$ we have

$$50\varepsilon \geq \int_{B_a} h(a,x)dx = \int_{a+B_a} h_1(y)dy \tag{4.7}$$

$$50\varepsilon \geq \int_{B_{a+b}} h(a+b,x)dx = \int_{a+b+B_{a+b}} h_2(y)dy \tag{4.8}$$

Since $B_a = [\max(0, 1/2-a), \min(1, 3/2-a)]$, for $0 \leq b \leq 1/2$ and $a+b \leq 1$ it is simple to see that $B_a \cap (b + B_{a+b})$ is an interval of length $\geq 1/2$. Since we assume $0 \leq b \leq 1/8, |c| \leq 1/8$ there exists an interval I of length $\geq 1/4$ such that for $t \in I$, we have

$$t, t+b, t-c, t+b-c \in (a+B_a) \cap a+b+B_{a+b} \subset a+J.$$

From (4.6), (4.4), we see that for $y, y-c \in a+J$ we have

$$\delta \leq g'(y) - f(y-c) = f(y) + h_1(y) - f(y-c).$$

The details of the following calculation differ depending on the sign of c. For definiteness, we assume $c \geq 0$. We have, for $t \in I$,

$$\begin{aligned}
\delta b = \int_t^{t+b} \delta dy &\leq \int_t^{t+b} (f(y) - f(y-c) + h_1(y))dy \\
&= \int_t^{t+b} f(y)dy - \int_{t-c}^{t+b-c} f(y)dy + \int_t^{t+b} h_1(y)dy \\
&= \int_{t+b-c}^{t+b} f(y)dy - \int_{t-c}^{t} f(y)dy + \int_t^{t+b} h_1(y)dy \\
&= \int_{t+b-c}^{t+b} (f(y) - f(y-b))dy + \int_t^{t+b} h_1(y)dy.
\end{aligned}$$

From (4.4), (4.5), we have

$$f(y) = g'(y) - h_1(y) = f(y-b) + h_2(y) - h_1(y) \\ \leq f(y-b) + h_2(y)$$

and thus

$$\delta b \leq \int_{t+b-c}^{t+b} h_2(y)dy + \int_{t}^{t+b} h_1(y)dy.$$

We integrate this inequality for $t \in I$. Since I has length $\geq 1/4$, we get

$$\frac{\delta b}{4} \leq \int_{t\in I}\int_{y\in[t+b-c,t+b]} h_2(y)dtdy + \int_{t\in I}\int_{y\in[t,t+b]} h_1(y)dtdy. \tag{4.9}$$

We observe that for $t \in I$, we have $[t, t+b] \subset a + B_a$. Using Fubini Theorem yields

$$\int_{t\in I}\int_{y\in[t,t+b]} h_1(y)dtdy \leq b \int_{a+B_a} h_1(y)dy \leq 50\varepsilon b.$$

A similar treatment of the first term of (4.9) yields

$$\frac{\delta b}{4} \leq 50\varepsilon(c+b), \text{ i.e. } \delta \leq 200\varepsilon(1+c/b).$$

∎

The technical restriction $|c| \leq 1/8$ in Lemma 4.1 prevents this lemma to provide global information on the behavior of f on J, although it does give information on large subintervals. The purpose of the next lemma is to patch these informations together, to obtain information on the whole of J.

Lemma 4.2. *We can find a_1, α_1 with the following properties.*
(4.10) *For $x \in J, f_1(x) \leq a_1 + \alpha_1 x + K\varepsilon$*
(4.11) *For $x \in U, f_1(x) \geq a_1 + \alpha_1 x$.*

Proof. For $i = 1, \ldots, 21$, the interval $[(2i-2)/41, (2i-1)/41]$ is of length $1/41$. Since $\lambda(U) \geq 49/50$, we can find $u_i \in U$ in this interval. We observe that $1/41 \leq u_{i+1} - u_i \leq 3/41 \leq 1/8$.

For $i = 1, \ldots, 20$, we consider the equation $b_i + \beta_i x$ of the line through $(u_i, f_1(u_i))$ and $(u_{i+1}, f_1(u_{i+1}))$. Consider now $x \in J, |x - u_i| \leq 1/8$. It follows from Lemma 4.1, used with $a = u_i, a+b = u_{i+1}, a+c = x$ that

$$f_1(x) \leq b_i + \beta_i x + K\varepsilon. \tag{4.12}$$

Fix $1 \leq i \leq 19$, and let $\xi(x) = b + \beta x$ be the equation of the line through $(u_i, f_1(u_i))$ and $(u_{i+2}, f_1(u_{i+2}))$. Observe that $u_{i+2} - u_i \leq 5/41 \leq 1/8$. Using Lemma 4.1 with $a = u_i, a+b = u_{i+2}, a+c = u_{i+1}$ yields

$$f_1(u_{i+1}) \leq \xi(u_{i+1}) + K\varepsilon.$$

Now

$$\begin{aligned} f_1(u_{i+1}) &= f_1(u_i) + \beta_i(u_{i+1} - u_i) \\ &= f_1(u_i) + \beta(u_{i+1} - u_i) + (\beta_i - \beta)(u_{i+1} - u_i) \\ &= \xi(u_{i+1}) + (\beta_i - \beta)(u_{i+1} - u_i) \le \xi(u_{i+1}) + K\varepsilon. \end{aligned}$$

Since $u_{i+1} - u_i \ge 1/41$, we see that $\beta_i - \beta \le K\varepsilon$. Using (4.12) for $x = u_{i+2}$, we have

$$\xi(u_{i+2}) = f_1(u_{i+2}) \le b_i + \beta_i u_{i+2} + K\varepsilon.$$

Since $\xi(u_i) = f_1(u_i) = b_i + \beta_i u_i$, we get

$$\beta(u_{i+2} - u_i) = \xi(u_{i+2}) - \xi(u_i) \le \beta_i(u_{i+2} - u_i) + K\varepsilon$$

and hence $\beta - \beta_i \le K\varepsilon$. This shows that $|\beta_i - \beta| \le K\varepsilon$. A similar argument shows that $|\beta - \beta_{i+1}| \le K\varepsilon$, and thus $|\beta_i - \beta_{i+1}| \le K\varepsilon$. Thus we see that $|\beta_i - \beta_{10}| \le K\varepsilon$ for $i \le 20$.

We observe that

$$b_i + \beta_i u_{i+1} = f(u_{i+1}) = b_{i+1} + \beta_{i+1} u_{i+1}$$

so that $|b_{i+1} - b_i| \le |\beta_{i+1} - \beta_i| \le K\varepsilon$. Thus $|b_i - b_{10}| \le K\varepsilon$ for $i \le 20$.

Given $x \in J$, we can find i such that $|x - u_i| \le 1/8$ (here we use that $1/10 + 1/41 \le 1/8$). From (4.12) we have

$$f_1(x) \le b_i + \beta_i x + K\varepsilon$$

so that $f_1(x) \le b_{10} + \beta_{10} x + K\varepsilon$. In order to prove (4.10), (4.11), it is thus enough to show that $f_1(x) \ge b_{10} + \beta_{10} x - K\varepsilon$ for $x \in U$ (we then set $a_1 = b_{10} - K\varepsilon, \alpha_1 = \beta_{10}$). Suppose first that $u_i < x \le u_{i+1}$ for some $1 \le i \le 20$. Then if $b' + \beta' x$ denotes the equation of the line through $(u_i, f_1(u_i))$ and $(x, f_1(x))$, we can use Lemma 4.2 to see that

$$\begin{aligned} f_1(u_{i+1}) &\le b' + \beta' u_{i+1} + K\varepsilon \\ &= f_1(u_i) + \beta'(u_{i+1} - u_i) + K\varepsilon \\ &= f_1(u_{i+1}) + (\beta' - \beta_i)(u_{i+1} - u_i) + K\varepsilon. \end{aligned}$$

It follows that $\beta' \ge \beta_i - K\varepsilon$. Thus

$$\begin{aligned} f_1(x) &= f_1(u_i) + \beta'(x - u_i) \ge f_1(u_i) + \beta_i(x - u_i) - K\varepsilon \\ &= b_i + \beta_i x - K\varepsilon \ge b_{10} + \beta_{10} x - K\varepsilon. \end{aligned}$$

We now consider the case $x \le u_1$ (the case $x > u_2$ is similar and is left to the reader). We recall that

$$f_1(u_1) = b_1 + \beta_1 u_1 \ge b_{10} + \beta_{10} u_1 - K\varepsilon. \tag{4.13}$$

The values of b_{10}, β_{10} do not depend on the choice of u_1; so (4.13) still hold when u_1 is replaced by x. This finishes the proof. ∎

Lemma 4.3. *We can find α, a_1, a_2 with the following properties*

(4.14) For $x \in J, i = 1, 2,\ f_i(x) \le a_i + \alpha x + K\varepsilon$

(4.15) For $x \in U,\ i = 1, 2,\ f_i(x) \ge a_i + \alpha x$.

Proof. Lemma 4.2 says that we can achieve this where α is replaced by a number α_i depending on i. It clearly then suffices to show that $|\alpha_1-\alpha_2|\le K\varepsilon$. There is no loss of generality to assume $\alpha_1\ge\alpha_2$.

Consider $x_1\in[0,1/4], x_2\in[3/4,1]$. We have

$$\begin{aligned}f_1(x_1)+f_2(x_2) &\le a_1+\alpha_1x_1+K\varepsilon+a_2+\alpha_2x_2+K\varepsilon\\ &= a_1+a_2+(\alpha_1-\alpha_2)x_1+\alpha_2(x_1+x_2)+2K\varepsilon\\ &\le a_1+a_2+\frac{1}{4}(\alpha_1-\alpha_2)+\alpha_2(x_1+x_2)+2K\varepsilon.\end{aligned}\tag{4.16}$$

We note that $|x_1+x_2-1|\le 1/4$. Since $\lambda([0,1]\setminus U)\le 1/50$, it is clear that we can find $x\in U$, $x\ge 1/2$, such that $x_1+x_2-x\in U$. We then have

$$\begin{aligned}g(x_1+x_2) &\ge f_1(x)+f_2(x_1+x_2-x)\\ &\ge a_1+a_2+\alpha_1x+\alpha_2(x_1+x_2-x)\\ &= a_1+a_2+\alpha_2(x_1+x_2)+(\alpha_1-\alpha_2)x\\ &\ge a_1+a_2+\alpha_2(x_1+x_2)+\frac{1}{2}(\alpha_1-\alpha_2).\end{aligned}\tag{4.17}$$

From (4.16) and (4.17) follows that

$$h(x_1,x_2)=g(x_1+x_2)-f_1(x_1)-f_2(x_2)\ge\frac{1}{4}(\alpha_1-\alpha_2)-2K\varepsilon.$$

If we integrate this inequality over $x_1\in[0,1/4], x_2\in[3/4,1]$, we get $\alpha_1-\alpha_2\le K\varepsilon$. ∎

We are now ready to prove Proposition 3.3. Consider a_i,α,K as in (4.14), (4.15). Set, for $i=1,2$,

$$f_i'(x)=\mathrm{Max}(f_i(x),\ a_i+\alpha x-K\varepsilon).$$

Thus $|f_i'(x)-(a_i+\alpha x)|\le K\varepsilon$. Thus it suffices to show that for $x_1,x_2\in J, x_1+x_2\in\left[\frac{4}{10},\frac{16}{10}\right]$, we have $f_1'(x_1)+f_2'(x_2)\le g(x_1+x_2)$. This holds if $f_1'(x_1)=f_1(x_1)$ and $f_2'(x_2)=f_2(x_2)$. Otherwise $f_1'(x_1)+f_2'(x_2)\le a_1+a_2+\alpha(x_1+x_2)$. Clearly there exists $y_1\in U, y_2\in U$ such that $y_1+y_2=x_1+x_2$. Thus

$$g(x_1+x_2)=g(y_1+y_2)\ge f_1(y_1)+f_2(y_2)\ge a_1+\alpha y_1+a_2+\alpha y_2=a_1+a_2+\alpha(x_1+x_2).$$

Proposition 3.3 is proved.

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